Field-induced drift and trapping in percolation networks

1984 J. Phys. A: Math. Gen. 17 2995

(http://iopscience.iop.org/0305-4470/17/15/017)

J. Phys. A: Math. Gen. 17 (1984) 2995–3008. Printed in Great Britain

Field-induced drift and trapping in percolation networks

Steven R White and Mustansir Barma†
Department of Physics and Baker Laboratory, Cornell University, Ithaca, NY 14853, USA

Received 10 April 1984

Abstract. We study the effects of a bias-producing external field on a random walk on the infinite cluster in the percolation problem. There are two competing physical effects (drift and trapping) which result in a drift velocity v which rises and then falls as the field increases. We study these effects on a one-dimensional lattice with random-length branches, and on the diluted Bethe lattice. We calculate v in the first model with a maximum allowed length for each branch. In the limit that this cutoff length becomes infinite, we find that the velocity vanishes identically above a finite critical value of the field. For the Bethe lattice, we derive an upper bound on the critical field, which varies as $(p-p_c)^{1/2}$ as the percolation concentration p_c is approached. In the one-dimensional model, we also investigate the anomalous regime in which the velocity vanishes. We discuss the distribution of steady state times required to traverse N sites, and find that it can be described in terms of a stable distribution of index x with superimposed oscillations. The index x of the stable distribution is given by L/ξ where ξ is a characteristic branch length and L is a bias-induced length which describes the exponential buildup of the steady state density of particles towards the end of a branch.

1. Introduction

The problem of a random walker in an infinite percolation cluster—the 'ant in the labyrinth' (de Gennes 1976)—is of great current interest, and a good deal is known not only about the behaviour at very long times, but also about the crossover that occurs when the typical distance traversed matches the correlation length (Gefen *et al* 1983, Pandey *et al* 1984). Also very interesting, but not nearly as well understood, is the effect of an external field which produces a bias in the random walk, making the ant more likely to step along the field than against it. The bias has a dual effect: it induces drift in the direction of the field, but also creates traps, such as in dead-end branches, to escape from which the ant must move against the field. In the large-field limit in which the ant is not allowed to step against the field, there is no macroscopic drift (Vicsek *et al* 1982). Thus one finds (Böttger and Bryskin 1982, Barma and Dhar 1983, Pandey 1984) that the dirft velocity v is not a monotonic function of the bais (as it is for a non-random medium). Monte Carlo simulations (Pandey 1984) show that the average distance moved by the ant in a given time is a non-monotonic function of the bias. Furthermore, it has been argued (Barma and Dhar 1983) that the drift velocity vanishes once the bias exceeds a finite threshold value. Dhar (1984) has argued that when $v=0$, the typical distance covered by a particle grows as a power of the time with an exponent less than unity.

† On leave from Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India.

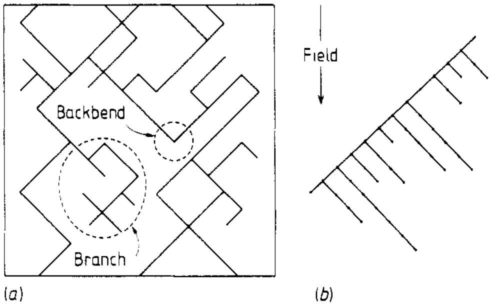

Figure 1. (*a*) Part of the infinite percolation cluster. Two types of traps are shown: a branch predominantly in the direction of the field, and a backbend in the backbone. (*b*) The random comb: a diffusing particle may be trapped in the branches of random lengths for a long time.

In this paper, we study two models which exhibit field-induced trapping. The first model is one-dimensional in character, and illustrates how the trapping mechanism works. It is defined on a 'random comb' (figure 1) which consists of a one-dimensional lattice with random length branches, with a maximum allowable length M for each branch. We calculate the drift velocity as a function of the bias, and in the limit $M \to \infty$, we find that the velocity v vanishes above a finite critical value of the field. We also study the distribution, over the ensemble of combs, of steady state times required to traverse a given length of the comb. In the anomalous $v = 0$ regime, we find that this distribution can be described in terms of the stable distribution with index $x < 1$ (Feller 1971), but with superimposed oscillations. The index x is given by the ratio $L(g)/\xi$ of the two fundamental lengths in the problem. The length ξ enters into the distribution of branch lengths (1), whereas $L(g)$ is a bias-induced length, defined in (11), which describes the exponential buildup of the steady state density of particles towards the end of a branch.

The second model we consider is that of a biased random walk on a randomly diluted Bethe lattice (§ 3). In this case, we cannot solve explicitly for v, but we do find that branch trapping causes the drift velocity to vanish once the field exceeds a finite value which depends on the concentration of sites p. This upper bound on the critical field vanishes as $(p - p_c)^{1/2}$ as p approaches the percolation concentration p_c.

An anomalous regime in which the drift velocity v vanishes has been found in other models too. Scher and Montroll (1975) and Shlesinger (1974) have discussed transport in the context of continuous time random walks with a distribution of pausing times. If this distribution decays as a slow power law, one finds that $v = 0$. Another model which exhibits an anomalous regime consists of a chain in which the direction of bias of each bond is a random variable (Kesten *et al* 1975, Derrida 1983 and references therein). For sufficiently large values of the bias, one finds that the drift velocity vanishes.

2. Branch trapping on a random comb

The physical origin of the non-monotonic behaviour of the drift velocity in a biased random walk on an infinite percolation cluster is that the external field, in conjunction with the random geometry of the cluster, sets up traps in which the particle gets localised for a long time. These traps are of two sorts—branches off the backbone which point predominantly along the field, and backbends in the backbone (those regions of the backbone from which a particle can escape only by travelling against the field). Both types of traps are illustrated in figure 1(a).

Here we discuss branch trapping on a random comb, which is illustrated in figure 1(b). It consists of a linear chain (the backbone) of sites evenly spaced a distance a apart. From each backbone site emanates a finite linear chain (a 'branch') of sites, also with spacing a. All branches are taken to run in the direction of the field, as is the backbone. Periodic boundary conditions are assumed for the backbone. We take the length l_i of the branch attached to site i on the backbone to be a discrete, independent random variable distributed with probabilities

$$p(l_i) = \frac{1-\exp(-a/\xi)}{1-\exp[-(M+1)a/\xi]}\,\mathrm{e}^{-l_i/\xi}, \qquad l_i = 0, a, 2a, \ldots, Ma. \tag{1}$$

In order to ensure that the system is finite, we have imposed a cutoff, Ma, on the maximum possible length of each branch, but we will eventually consider the limit $M\to\infty$. The random comb has some resemblance to a model discussed by Ziman (1979) in the context of the propagation of excitations on percolation networks, but there is a crucial difference: in the Ziman model, all branches have equal lengths, whereas in our case there is a finite probability of having any branch length from 0 to Ma. It is the possibility of having indefinitely long branches in the limit $M\to\infty$ which is responsible for the anomalous behaviour associated with the vanishing of the drift velocity.

Consider a particular realisation $\boldsymbol{R}$ of the random comb, which we specify by the set of branch lengths $l_1, l_2, \ldots, l_N$. We assume that a particle placed on any site of the random comb performs a random walk with the end of each branch acting like a reflecting barrier. The effect of the field is modeled as a bias in the random walk, making steps along the field more likely than against it. If $u_n(t)$ is the probability that a particle is at site n at time t, we have

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = \sum_m (W_{mn}u_m - W_{nm}u_n) \tag{2}$$

where the summation runs over the nearest neighbours of site n. The transition probability per unit time W_{mn} to hop from site m to site n is given by

$$W_{mn} = W(1\pm g) \tag{3}$$

where the bias g satisfies $0\leq g<1$, and the plus (minus) sign is for hops along (against) the direction of the field. Equation (2) can be rewritten in matrix form as

$$(\mathrm{d}/\mathrm{d}t)|u(t;\boldsymbol{R})\rangle = \boldsymbol{W}(\boldsymbol{R})|u(t;\boldsymbol{R})\rangle. \tag{4}$$

Here $|u(t;\boldsymbol{R})\rangle$ denotes the column vector with entries $u_n(t)$ where n runs over all sites in realisation $\boldsymbol{R}$ of the random comb. Both the size and the entries of the matrix $\boldsymbol{W}(\boldsymbol{R})$ depend on the realisation $\boldsymbol{R}$. The formal solution of (4) is

$$|u(t;\boldsymbol{R})\rangle = \exp[\boldsymbol{W}(\boldsymbol{R})t]|u(0;\boldsymbol{R})\rangle \tag{5}$$

and can be used to find quantities of physical interest, which can then be averaged over all configurations $\boldsymbol{R}$. For instance, the configuration averaged probability of being on site n of the backbone at time t is

$$\sum_{\boldsymbol{R}} P(\boldsymbol{R})\langle n|u(t;\boldsymbol{R})\rangle \tag{6}$$

where $P(\boldsymbol{R})$, the probability of occurrence of realisation $\boldsymbol{R}$, is given by

$$P(\boldsymbol{R}) = \prod_{i=1}^{N} p(l_i). \tag{7}$$

We have not obtained the full time-dependent solution embodied, for instance, in (6). Rather we have studied the properties of the system in the steady state. Since the system is finite and every site can be reached from every other, we expect that the steady state will be reached as $t \to \infty$, irrespective of the initial conditions (Mathews *et al* 1960). The steady state distribution of a set of non-interacting particles on the random comb is found by setting the left-hand side of (4) to zero. It is given, up to a multiplicative constant, by the components of the right eigenvector of $\boldsymbol{W}(\boldsymbol{R})$ corresponding to eigenvalue 0.

Let ρ_m be the steady state density at site m, i.e., the number of particles on that site in steady state. Note that ρ_m is the product of the number of particles $\mathcal{N}$ in the system and the value of u_m in the steady state. The net current flowing from site m to an adjacent site n is given by

$$J_{mn} = W_{mn}\rho_m - W_{nm}\rho_n. \tag{8}$$

In steady state, the branches carry no current and hence have no effect on the density of the backbone. The periodic boundary conditions ensure that the backbone density ρ_0 is the same at all backbone sites, and using (8) we find

$$\rho_0 = J/2Wg \tag{9}$$

where J is the current on the backbone. The density ρ_l on a branch site at distance l from the backbone is found, on setting J_{mn} to zero for each link in the branch, to be given by

$$\rho_l = \rho_0 \, e^{l/L(g)} \tag{10}$$

where $L(g)$ is a new length introduced into the problem by the bias and is given by

$$L(g) = a\{\ln[(1+g)/(1-g)]\}^{-1}. \tag{11}$$

Let us define a steady state transit time for the N-site backbone associated with realisation $\boldsymbol{R}$ as

$$T_N(\boldsymbol{R}) = \mathcal{N}/J \tag{12}$$

where $\mathcal{N}$ is the total number of particles in the system and J is the backbone current. We expect $T_N(\boldsymbol{R})$ to be the mean time of traversal of a particle through the system. We can see this as follows: imagine the backbone to be in the shape of a ring and consider the motion of a single particle on it. Let $w(t)$ be the *net* number of traversals made by the particle in the direction of the field in time t. The mean traversal time is then

$$\bar{T}_N(\boldsymbol{R}) = \lim_{t\to\infty} t/w(t). \tag{13}$$

The contribution of this single particle to the mean current between two backbone sites is

$$\bar{J}_1 = \lim_{t\to\infty} w(t)/t. \tag{14}$$

If there are $\mathcal{N}$ non-interacting particles, the mean current $\bar{J}$ is given by $\mathcal{N}\bar{J}_1$. We thus have

$$\bar{T}_N(\boldsymbol{R}) = \mathcal{N}/\bar{J}. \tag{15}$$

If we assume the system is ergodic, we have $\bar{J} = J$, and in that case $T_N(\boldsymbol{R})$ equals the mean traversal time $\bar{T}_N(\boldsymbol{R})$.

Putting equations (9), (10), and (12) together we find

$$T_N(\boldsymbol{R}) = \sum_{i=1}^{N} t_i \tag{16}$$

where the sum is over backbone sites i and

$$t_i = (2Wg)^{-1} \frac{\exp[(l_i + a)/L(g)] - 1}{\exp[a/L(g)] - 1} \tag{17}$$

may be identified as the mean time spent by a particle on backbone site i and its associated branch of length l_i. Using the probability distribution (1) we find that the configuration average of $T_N(\boldsymbol{R})$ is

$$\begin{aligned} \langle T_N \rangle &\equiv \sum_{\boldsymbol{R}} P(\boldsymbol{R}) T_N(\boldsymbol{R}) \\ &= \frac{N}{2Wg} \frac{1}{1 - \exp[a/L(g)]} \left(1 - e^{a/L(g)} \frac{G_M[\exp(-a/\xi)]}{G_M\{\exp[a/L(g) - a/\xi]\}} \right) \end{aligned} \tag{18}$$

where the function $G_M(y)$ is given by

$$G_M(y) = (1-y)/(1-y^{M+1}). \tag{19}$$

We define the drift velocity v_M as

$$v_M \equiv Na/\langle T_N \rangle \tag{20}$$

and have plotted v_M as a function of g in figure 2 for various values of the branch cutoff parameter M. As $M \to \infty$, v_M approaches a limiting form, which we denote by v. We find

$$\begin{aligned} v &= 2Wga[1 - \exp(a/L(g) - a/\xi)] && \text{for } L(g) \geqslant \xi \\ &= 0 && \text{for } L(g) \leqslant \xi. \end{aligned} \tag{21}$$

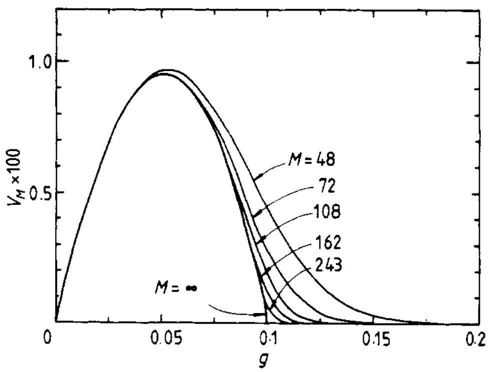

Figure 2. The drift velocity v_M measured in units of Wa of a particle on the random comb as a function of the bias g for various values of the branch cutoff length M, with $\xi = 5a$.

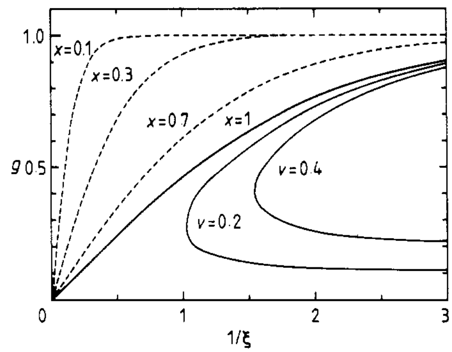

Figure 3. Phase diagram for the random comb. The bold line separates the drift regime from the anomalous regime. Curves of constant v are shown in the former, as are curves of constant x in the latter. The units of ξ and v are a and Wa, respectively.

Contours of equal limiting velocity v in the (g, ξ^{-1}) plane are shown in figure 3. It is of interest to ask for the deviations of v_M from v when M is large but finite. We find that as $M \to \infty$, the difference is

$$\begin{aligned} v_M - v &\sim \exp[-(M+1)a|L^{-1}(g) - \xi^{-1}|] && \text{if } L(g) \neq \xi \\ &\sim 1/M && \text{if } L(g) = \xi. \end{aligned} \tag{22}$$

Let us define x as the ratio of the two characteristic lengths in the problem

$$x = L(g)/\xi. \tag{23}$$

In the limit of low bias ($g \to 0$) with fixed $x > 1$, both $L(g)$ and ξ diverge. The velocity can then be written in the scaling form

$$v(L(g), \xi) = Wa^3\xi^{-2}Y(x) \tag{24}$$

where the scaling function $Y(x)$ is given by

$$Y(x) = (x-1)/x^2. \tag{25}$$

Consider now the question of the distribution over the ensemble of combs $\boldsymbol{R}$ of steady state transit times $T_N(\boldsymbol{R})$ required by a particle to traverse the N-site backbone. This question is particularly interesting when $x < 1$, so that $v = 0$ and $\langle T_N \rangle$ diverges in the limit $M \to \infty$. Let us first address the question in a slightly different model, in which the lengths of the branches are allowed to take on continuous values with probability density

$$\tilde{p}(l) = \xi^{-1}\,\mathrm{e}^{-l/\xi} \qquad 0 \leq l \leq \infty \tag{26}$$

where the tilde is used here and below to indicate continuous branch lengths. By using (17) and (26) and the relation $\tilde{P}(t) = \tilde{p}(l)\,\mathrm{d}l/\mathrm{d}t$ it is straightforward to deduce the probability density for the branch steady state times t_i. We find that

$$\tilde{P}(t_i) = bx\exp(a/\xi)(1+bt_i)^{-1-x} \qquad (2Wg)^{-1} \leq t_i < \infty \tag{27}$$

where b is given by

$$b = 4Wg^2(1-g)^{-1}. \tag{28}$$

The power law decay which characterises this density for large t_i becomes slower as the bias is increased with ξ held fixed, since by (11) and (23), x decreases with g.

The steady state transit time T_N is the sum of N identically distributed variables t_i, each described by the probability density (27). The behaviour of the probability distribution for T_N in the limit $N \to \infty$ has been studied for arbitrary $\tilde{P}(t_i)$ in probability theory (Feller 1971, Gnedenko and Kolmogorov 1954). Provided certain necessary and sufficient conditions are satisfied (Feller 1971, p 312) and T_N is rescaled by an appropriate function of N, one finds that the distribution approaches a well defined limiting form. The necessary conditions are, in fact, satisfied by the probability density (27), and one thus obtains a limiting form which depends on x. For $x > 2$, the variance σ^2 of the density (27) exists and the central limit theorem applies, with the result that as $N \to \infty$, $(T_N - N\langle t_i \rangle)/\sigma\sqrt{N}$ is normally distributed. For $x < 2$, however, the variance diverges, and the distribution for T_N is described in the limit $N \to \infty$ by a stable distribution of index x (Feller 1971). As long as x exceeds unity, the law of large numbers holds and the density for T_N/N is sharply peaked around $\langle t_i \rangle$. By contrast, once x falls below unity, the system enters an anomalous regime in which $\langle t_i \rangle$, and thus $\langle T_N \rangle$, diverges. In this case we have, in the limit $N \to \infty$,

$$\tilde{P}_N(T_N) \approx b(cN)^{-1/x} P_x(bT_N/(cN)^{1/x}) \tag{29}$$

where we have

$$c = \Gamma(1-x)\exp(a/\xi). \tag{30}$$

The stable distribution $P_x(y)$ is known in terms of simple functions only for $x = \frac{1}{2}$, although it has the simple Laplace transform

$$\mathscr{L}[P_x(y)] = \exp(-s^x) \tag{31}$$

for all $x < 1$ (Feller 1971). It is interesting to note that the one-dimensional random bias model studied by Kesten *et al* (1975) has a distribution of traversal times which approaches a stable distribution.

From (29) we see that the probability density $\tilde{P}_N(T_N)$ approaches a scaling function of T_N and N. If we identify a 'typical' value T_N^*, say by the location of the maximum of $\tilde{P}_N(T_N)$, we see from (29) that

$$T_N^* \propto N^{1/x}. \tag{32}$$

The 'typical' steady state time T_N^* grows faster than linearly with N, consistent with the vanishing of the drift velocity. Curves of constant x in the (g, ξ^{-1}) plane are shown in figure 3.

Returning to the discrete density (1), we note that its integrated probability distribution coincides at all integer values with that of the continuous density (26). Thus we might expect the resulting probability density $P_N(T_N)$ to be similar to $\tilde{P}_N(T_N)$. But on using the theorem for the approach to a stable distribution (Feller 1971, p 312) we find that $P_N(T_N)$ does *not* tend to a limiting distribution as $N \to \infty$.

In order to investigate the failure of $P_N(T_N)$ to approach a stable distribution we have numerically determined the probability density $P_N(T_N)$ for several values of N and x. We used a Monte Carlo method in which many realisations $\boldsymbol{R}$ of the comb were generated according to (1), and $T_N(\boldsymbol{R})$ calculated for each $\boldsymbol{R}$, using (16) and (17). Figure 4 shows schematic results for $x = \frac{1}{2}$ and $N = 64$ and 512, along with the analytically known (Feller 1971, p 173) stable curve $\tilde{P}_{x=1/2}(T_N)$. The scale on the time axis for the $N = 512$ curve was matched with the analytic curve, while the time

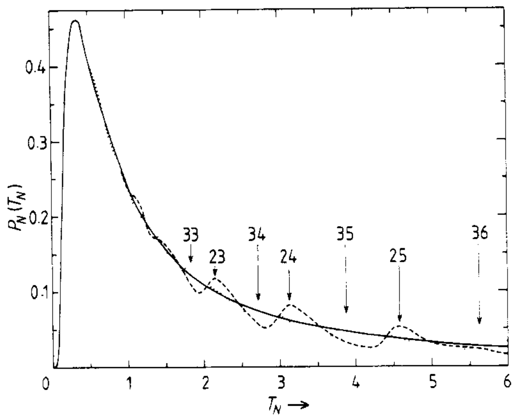

Figure 4. The probability densities $P_N(T_N)$ for traversal times on the random comb for $N=64$ (----) and $N=512$ (······) are displayed for the case $x=\frac{1}{2}$, along with the corresponding stable density (———). The parameters $L(g)$ and ξ were $2.5a$ and $5a$, respectively. The horizontal scale for the $N=512$ curve is 2.58×10^6 and for the $N=64$ curve is 4.04×10^4, in units of W^{-1}. Each bump is labelled by the number of sites in the long branch which produces it.

axis scale for $N=64$ was taken to be the scale for $N=512$ divided by $(512/64)^{1/x}=64$. We see that the curves nearly coincide with the stable density, except that for fairly long times the probability density is concentrated in bumps, with the spacing between bumps growing roughly geometrically. Similar geometrically spaced oscillations have been found in the configuration-averaged mean displacement of a particle in a one-dimensional model considered by Bernasconi and Schneider (1982). We have been able to interpret each bump in figure 4 as arising from the presence, in some realisations, of a single very long branch of length l in which a particle spends a much longer time than in the rest of the system. This interpretation allows one to label each bump by l, as has been done in the figure. If $t(l)$ is the mean time spent in a branch of length l, the shape of each bump is approximately given by $P_{N-1}(T_N-t(l))$ times the probability of occurrence of a branch of length l. The bumps shift inward on rescaling T_N, and since the spacing between them increases geometrically, the number of bumps in any interval can at most fluctuate between two successive integers.

Although we have not been able to find an explicit formula for the probability density $P_N(T_N)$ for discrete chains, we can derive bounds for the corresponding cumulative distribution $F_N(T_N)$ defined by

$$F_N(T_N)=\int_0^{T_N} \mathrm{d}T\, P_N(T). \qquad (33)$$

To this end we note first that the discrete distribution of branch lengths (2) gives rise (in the limit $M\to\infty$) to the cumulative distribution

$$f(l)=[1-\exp(-a/\xi)]\sum_{m=0}^{\infty} \mathrm{e}^{-ma/\xi}\,\theta(l-ma) \qquad (34)$$

where $\theta(z)$ is the step function.

Let the cumulative probability distribution corresponding to the continuous probability density (26) be $\tilde{f}(l)$. We have

$$\tilde{f}(l)=1-\mathrm{e}^{-l/\xi}, \qquad 0\leq l<\infty. \qquad (35)$$

Finally, let us define another cumulative distribution

$$\tilde{f}'(l) = 1 - e^{-(l+a)/\xi}, \qquad 0 \leq l < \infty. \tag{36}$$

Then one can verify the relations

$$\tilde{f}(l) \leq f(l) \leq \tilde{f}'(l). \tag{37}$$

Now (17) gives us a relation between l and t (we have dropped the subscript i for the present). Note that t is a monotonic function of l, and so the mapping is invertible. Equation (17), in conjunction with (34), (35) and (36), can be used to deduce the corresponding cumulative distribution in terms of t, namely $F(t)$, $\tilde{F}(t)$ and $\tilde{F}'(t)$. Explicitly, we have

$$F(t) = f(l(t)) \tag{38}$$

etc. Consequently, on using (37), we find

$$\tilde{F}(t) \leq F(t) \leq \tilde{F}'(t). \tag{39}$$

Now T_N is the sum of N t_i's (equation 16), and (39) holds for each t_i. It then follows for the cumulative distribution functions for T_N that

$$\tilde{F}_N(T_N) \leq F_N(T_N) \leq \tilde{F}'_N(T_N). \tag{40}$$

Equation (40) provides fairly stringent bounds on $F_N(T_N)$ because $\tilde{F}_N(bT_N/[cN]^{1/x})$ and $\tilde{F}'_N(bT_N/[\Gamma(1-x)N]^{1/x})$ tend to the same stable distribution as $N \to \infty$.

3. The Bethe lattice

In the model of § 2, the backbone and each branch were taken to be linear and were only allowed to point along the field. Neither of these features is particularly realistic, and in this section we consider the problem on a diluted Bethe lattice (Fisher and Essam 1961). We will show that the drift velocity v vanishes above a threshold value of the bias, although we cannot solve explicitly for v when it is non-zero.

At the outset, we must specify the 'easy' direction of hopping on each bond in the undiluted lattice. It might seem simplest to choose the easy direction to always point away from a particular root site (see figure 5(a)), but this would lead to fewer incoming than outgoing directions at every site. Except for briefly mentioning results for such a rooted model at the end of the section, we will consider an assignment† of directions

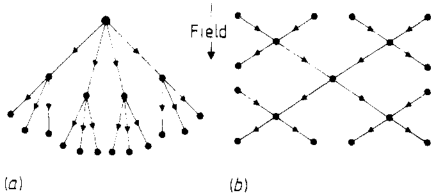

Figure 5. Two possible choices for the 'easy' hopping direction on the bonds of a Bethe lattice. (a) The easy direction always points away from a particular root site (the larger dot). (b) At each site, the easy direction points into the site for half the bonds, and out for the other half.

† This construction was suggested by Dhar; see also Straley (1977).

which resembles that on a finite-dimensional lattice: on each site of a tree of even coordination number, say $2m$, let the easy direction (the 'down' as opposed to the 'up' direction) point into the site for half the bounds, and out of the site the other half. For $m=2$ (see figure 5(b)), the tree resembles a square lattice locally (with the field along a diagonal), and for $m=3$ a cubic lattice, (with the field along a body diagonal). A feature of this construction is that it allows one to define a (dimensionless) 'vertical' displacement r_{ij} in the direction of the field. Explicitly, r_{ij} is the number of up steps *minus* the number of down steps taken on the unique chain of sites connecting i to j.

Now consider removing a fraction $1-p$ of the sites. Suppose $p>p_c=1/(2m-1)$ that there are infinite clusters (Fisher and Essam 1961). A point is said to be on the backbone of an infinite cluster if there are at least two non-overlapping paths connecting it to infinity. We define branches as follows: imagine removing all points of the backbone from the infinite cluster. One is left with finite, connected clusters of sites. Each such connected cluster is called a branch. An 'up' ('down') branch is one in which the first step from the backbone onto the branch is in the up (down) direction. Note that up to $2m-2$ branches may connect to the backbone at a particular backbone site.

Let a current J be injected into a site on the backbone of the infinite cluster, and in the steady state be collected at all the sites at infinity. Suppose ρ_i is the steady state number of particles on backbone site i to which is attached a branch β. Let us define $f_{i\beta}$ by letting $\rho_i f_{i\beta}$ be the steady state number of particles in the branch. Note that $f_{i\beta}$ must depend only on the configuration of the branch and the bias g. In the steady state no current flows through any bond in a branch. Thus, if j and j' are adjacent branch sites, with j above j' ($r_{jj'}=-1$), we obtain from (3)

$$\rho_{j'}/\rho_j = e^{a/L(g)} \tag{41}$$

where $L(g)$ was defined in (11). Hence, it follows that

$$f_{i\beta} = \sum_{j\in\beta} e^{-ar_{ij}/L(g)}. \tag{42}$$

The key simplifying feature of the Bethe lattice is that the possible configurations of a particular branch depend only on whether it is an up or a down branch. Let $\langle f_\pm \rangle$ be the average of $f_{i\beta}$ over all up (+) or down (−) branches. Let η_{ij} be an indicator function which takes the value 1 if site j is part of branch β attached to the backbone at site i, and is zero otherwise. From (42) we have

$$\langle f_\pm(g, p)\rangle = \sum_j e^{-ar_{ij}/L(g)} \langle \eta_{ij} \rangle \tag{43}$$

where j runs over all sites. We evaluate $\langle f_\pm \rangle$ in the appendix by a transfer matrix method. The result is

$$\begin{aligned} \langle f_\pm(g, p)\rangle &= Q(p) \sum_{n=1}^{2} \frac{C_n^\pm(g)}{1-\lambda_n(g)p[Q(p)]^{2m-2}} && \text{if } g<g_0 \\ &= \infty && \text{if } g \geqslant g_0 \end{aligned} \tag{44}$$

where g_0 is the solution of $\lambda_1(g)p[Q(p)]^{2m-2}=1$, and where the functions $\lambda_n(g)$, $C_n^\pm(g)$ and $Q(p)$ are defined in the appendix.

The divergences of $\langle f_+ \rangle$ and $\langle f_- \rangle$ when the bias g exceeds g_0 imply that the branch-configuration averaged number of particles $\rho_i\langle f_{i\beta}\rangle$ in a branch β attached to backbone site i is infinite. This in turn means that for $g \geqslant g_0$ the average over branch configurations of the steady state time spent on a backbone site and its branches is infinite. What this means for a particular realisation of the system is that a typical particle will spend a vanishing fraction of its time on the backbone. As for the random comb, we expect that $g \geq g_0$ is an anomalous regime in which the drift velocity vanishes.

If g_c is the critical value of the bias g which signals the onset of the anomalous regime, then we have $g_c \leq g_0$. We expect $g_c = g_0$, but cannot exclude the possibility that the average of ρ_i over all backbone sites i diverges for yet lower values of the bias. It had been argued (Barma and Dhar 1983) that as $p \to p_c$, the critical value of the bias g_c is proportional to $(p - p_c)^{\nu}$ where ν is the standard correlation length exponent. For the Bethe lattice, we find that the upper bound g_0 on the critical bias g_c reduces, in the limit $p \to p_c$, to

$$g_0 \approx [(m-1)(2m-1)/2m](p-p_c)^{1/2}. \tag{45}$$

If we take $g_c = g_0$, this result is in accord with the prediction of Barma and Dhar, as one has $\nu = \frac{1}{2}$ on the Bethe lattice (Straley 1982).

To conclude this section, we mention results for a Bethe lattice model in which the easy direction always points away from a particular root site (figure 5(a)). In this case, the natural measure of the displacement between two sites is the number of bonds on the path connecting them, as opposed to the 'vertical displacement' r_{ij}. With this choice, one has $\nu = 1$ (Straley 1982). The calculation for this model follows that in the appendix, except that it is not necessary to introduce a transfer matrix. We find that there is an anomalous regime above a critical value of the bias, as for the model discussed above, but in this case the upper bound on the critical field is proportional to $(p - p_c)$, in contrast to $(p - p_c)^{1/2}$.

4. Conclusions

We have seen that the drift velocity vanishes above a critical value of the bias for two models of a random medium—the random comb (in the limit of the cutoff branch length $M \to \infty$), and the diluted Bethe lattice. For the random comb, which is essentially one-dimensional in character, we found that a central role is played by the ratio x of the bias induced length $L(g)$ to the characteristic branch length ξ. For $x > 1$, and in the limit $L(g), \xi \to \infty$, the velocity v can be written in scaling form with x the argument of the scaling function. For $x < 1$, we find that the drift velocity vanishes and the distribution of steady-state transit times T_N required to traverse N backbone sites is closely related to the stable probability distribution with index x.

It should be noted that the characteristic behaviour we have seen for the random comb—a finite velocity regime followed by an anomalous $v = 0$ regime as the bias is increased—depends crucially on the exponential form for the distribution of branch lengths, $p(l) \propto \exp(-l/\xi)$. For instance, if we take $p(l) \propto \exp[-(l/\xi)^{\alpha}]$, then for all values of g in the interval $0 < g < 1$, we find $v = 0$ if $0 < \alpha < 1$, while $v \neq 0$ if $\alpha > 1$. The choice of our distribution (1) is motivated by our expectation that in the percolation problem, the probability that a site j is on a branch connected to the backbone at site i varies as $\exp(-r_{ij}/\xi_c)$ for $r_{ij} \gg \xi_c$, where ξ_c is the correlation length. We expect the behaviour of biased random walks on an infinite percolation cluster in dimensions $d \geq 2$ to be qualitatively the same as for the random comb. In particular, if we define $\hat{x} = L(g)/\xi_c$, then we expect the velocity to scale as

$$v = \xi_c^{-k} \hat{Y}(\hat{x}). \tag{46}$$

This should reduce to the linear response formula

$$v = 2Dg \tag{47}$$

as $g \to 0$. Here D is the diffusion constant which vanishes as $\xi_c^{-(t-\beta)/\nu}$, where β and ν are the usual percolation exponents describing the behaviour of the infinite cluster and correlation length, and t is the conductivity exponent (Kirkpatrick 1973). Matching (46) and (47) we find

$$\hat{Y}(\hat{x}) \sim 1/\hat{x} \qquad \text{as} \qquad \hat{x} \to \infty, \tag{48}$$

and

$$k = 1 + (t - \beta)/\nu. \tag{49}$$

Notice that linear response theory is valid only as long as $\xi \ll L(g)$, a conclusion that was also reached by Ohtsuki and Keyes (1984).

We have not discussed backbend trapping at all in this paper. A one-dimensional model of this phenomenon was discussed by Barma and Dhar (1983); in fact, their model reduces to that studied by Derrida (1983). But the problem on the backbone of an infinite percolation cluster still remains open. Another interesting problem concerns the effect of interactions between particles—they would affect our results considerably. Finally, the full time-dependent solution for the random comb, embodied, for example, in (6), remains to be found.

Acknowledgments

We would like to thank Michael E Fisher for valuable discussions and for incisive comments on the manuscript. We would also like to acknowledge helpful discussions and correspondence with D Dhar, S Redner, S Solla and D Stauffer. We are grateful to R B Pandey for sending us a copy of his paper before publication. The support of the National Science Foundation, in part through a Graduate Fellowship to SRW, is gratefully acknowledged.

Note added in proof. Recent Monte Carlo simulations (Seifert and Suessenbach 1984) of a biased random walk on two- and three-dimensional percolation networks ($p > p_c$) seem to show that $d(\ln \bar{R})/d(\ln t)$ is an oscillating function of t, where $\bar{R}$ is the average distance moved by a particle in time t.

Appendix

In this appendix we evaluate the sum

$$\langle f_\pm \rangle = \sum_j e^{-ar_{ij}/L(g)} \langle \eta_{ij} \rangle \tag{A1}$$

where i is a backbone site, j runs over all sites, and $\langle \eta_{ij} \rangle$ gives the probability that site j is part of a given up (+) or down (−) branch β connected to the backbone at i. Let k be the nearest neighbour of site i which is on branch β. We say a site j is a potential β-site if the unique chain of sites connecting j to i contains k. Then if j is not a potential β-site, $\eta_{ij}=0$, while if j is a potential β-site, we have

$$\langle\eta_{ij}\rangle = p^N[Q(p)]^{N(2m-2)+1} \tag{A2}$$

where p is the probability of occupation of a site, N is the number of bonds connecting i to j, and $Q(p)$ is the probability that all paths in a given direction from a site known to be occupied are finite. One finds $Q(p)$ satisfies the equation (Essam 1972)

$$Q(p) = q + p[Q(p)]^{2m-1}. \tag{A3}$$

Equation (A2) follows from the observation that $\langle\eta_{ij}\rangle$ is the probability that the N sites on the path from i to j are occupied, times the probability that none of the $(2m-2)N+1$ free directions around the chain of N sites leads to infinity.

On the unique path from i to j, associate pseudospins with bonds: $s_k=+1$ ($s_k=-1$) if the bond is traversed in the up (down) direction. Then

$$N = \sum_k |s_k| \tag{A4}$$

and

$$r_{ij} = \sum_k s_k \tag{A5}$$

where k runs over the bonds linking i to j. Now the + or − in $\langle f_\pm\rangle$ just denotes the sign of s_1.

Given a sequence $\{s_k\}$, starting from i, the location of j is not uniquely determined: at step k, there are m ways of having $s_k = s_{k-1}$, and $(m-1)$ ways of having $s_k=-s_{k-1}$. Thus (A1) becomes

$$\langle f_{s_1}\rangle = \sum_{N=1}^{\infty} p^N[Q(p)]^{N(2m-2)+1} \sum_{\{s_2\ldots s_N\}} \prod_{k=1}^{N-1} n(s_k, s_{k+1}) \exp\left(-\sum_{i=1}^{N} as_i \Big/ L(g)\right)$$

where

$$n(s_k, s_{k+1}) = \begin{cases} m & \text{if } s_{k+1} = s_k \\ m-1 & \text{if } s_{k+1} = -s_k. \end{cases} \tag{A7}$$

We can evaluate the right-hand side of (A6) by introducing the transfer matrix

$$T = \begin{pmatrix} m\,\mathrm{e}^{-h} & m-1 \\ m-1 & m\,\mathrm{e}^{h} \end{pmatrix} \tag{A8}$$

where $h = a/L(g)$. We have

$$\langle f_\pm\rangle = Q(p) \sum_{N=1}^{\infty} (p[Q(p)]^{2m-2})^N \sum_{s_N=\pm1} \mathrm{e}^{-(s_1+s_N)/2} \langle s_1|T^{N-1}|s_N\rangle \tag{A9}$$

$$= Q(p) \sum_{N=1}^{\infty} (p[Q(p)]^{2m-2})^N (C_1^{\pm}\lambda_1^{N-1} + C_2^{\pm}\lambda_2^{N-1}) \tag{A10}$$

where λ_1 and λ_2 are the eigenvalues of T:

$$\lambda_n(g) = m\cosh h \pm [m^2\sinh^2 h + (m-1)^2]^{1/2}. \tag{A11}$$

The + sign goes with $n=1$. The matrix elements are given by

$$C_n^{\pm} = \frac{(m-1)(\lambda_n - \mathrm{e}^{\mp h})}{2(\lambda_n - m\cosh h)(\lambda_n - m\,\mathrm{e}^{\mp h})}. \tag{A12}$$

If $pQ(p)^{2m-2}\lambda_1 < 1$, the sum in (A10) converges and we have

$$\langle f_\pm \rangle = Q(p) \sum_{n=1}^{2} \frac{C_n^\pm(g)}{1-\lambda_n(g)p[Q(p)]^{2m-2}}. \tag{A13}$$

References

Barma M and Dhar D 1983 *J. Phys. C: Solid State Phys.* **16** 1451

Bernasconi J and Schneider W R 1982 *J. Phys. A: Math. Gen.* **15** L729

Böttger H and Bryskin V V 1982 *Phys. Status Solidi* b **113** 9

de Gennes P G 1976 *Recherche* **7** 919

Derrida B 1983 *J Stat. Phys.* **31** 433

Dhar D 1984 *J. Phys. A: Math. Gen.* **17** L257

Essam J W 1972 *Phase Transitions and Critical Phenomena* vol 2, ed C Domb and M S Green (London: Academic)

Feller W 1971 *An Introduction to Probability Theory and Its Applications* vol 2 (New York: Wiley)

Fisher M E and Essam J W 1961 *J. Math. Phys.* **2** 609

Gefen Y, Alexander S and Aharony A 1983 *Phys. Rev. Lett.* **50** 77

Gnedenko B V and Kolmgorov A N 1954 *Limit Distributions for Sums of Independent Random Variables* (Cambridge, Mass.: Addison-Wesley)

Kesten H, Kozlov M V and Spitzer F 1975 *Composito Math.* **30** 145

Kirkpatrick S 1973 *Rev. Mod. Phys.* **45** 574

Mathews P M, Shapiro I I and Falkoff D L 1960 *Phys. Rev.* **120** 1

Ohtsuki T and Keyes T 1984 *Phys. Rev. Lett.* **52** 1177

Pandey R B 1984 *Phys. Rev.* B **30** 489

Pandey R B, Stauffer D, Margolina A and Zabolitzky J G 1984 *J. Stat. Phys.* **34** 427

Scher H and Montroll E W 1975 *Phys. Rev.* B **12** 2455

Seifert E and Suessenbach M 1984 *Preprint Cologne University*

Shlesinger M F 1974 *J. Stat. Phys.* **10** 421

Straley J P 1977 *J. Phys. C: Solid State Phys.* **10** 3009

—— 1982 *J. Phys. C: Solid State Phys.* **15** 2333

Vicsek T, Kertész J and Cserti J 1982 *J. Phys. A: Math. Gen.* **15** L189

Ziman T A L 1979 *J. Phys. C: Solid State Phys.* **12** 2645